Lecture notes on risk management, public policy, and the financial system

Market risk measurement in practice

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Nonlinearity in market risk

Market risk measurement for options

Portfolio VaR

Market risk in insurance

Nonlinearity in market risk
Nonlinearity and convexity
Delta-gamma and option risk

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Nonlinearity in market risk

- **Nonlinearity:** P&L or payoff of a security doesn't respond proportionally to risk factor returns
- **Examples** of securities with nonlinear payoffs:
  - Options: **gamma risk**
  - Bonds: **convexity risk** in non-callable as well as callable coupon and zero-coupon, mortgage-backed securities
- Security value $f(S_t)$ a function of risk factor S_t,
- $f(S_t)$ has second and nonzero higher derivatives$\rightarrow$
  - Large-magnitude returns have a proportionally larger or smaller P&L impact than small returns
  - $\rightarrow$Price changes in one direction may have a larger P&L impact than changes in the opposite direction

VaR techniques for nonlinear positions

Simulation with full repricing using asset valuation model, e.g. Black-Scholes formula

- Can use Monte Carlo or historical simulation of **underlying price** or risk factor returns
- But revaluation of position at each simulated return may itself require "expensive" simulations

Delta-gamma using linear-quadratic approximation of P&L responses to risk factor returns

- Trades accuracy for speed
- Tractable and quite accurate in many cases
- But may be inaccurate for some portfolios
- Can be combined with Monte Carlo or historical simulation of risk factor returns

Option risk and the "greeks"

- Option risk stemming from underlying asset price risk is nonlinear
  - Price risk of the underlying asset (→**delta**, **gamma**)
  - Sensitivity to underlying price greatest near strike, may fall off rapidly in- or out-of-the-money
  - →Apply delta-gamma, with $f(S_t)$ representing option price or value
- Options are exposed to other risk factors, including
  - Interest-rate risk or **rho**, since an option matures at a future date
  - Implied volatility or **vega** risk
- Options have **time value** that decays over time at a rate **theta**
  - Theta is not a risk, but a deterministic quantity
  - Depends on interest rates, implied volatility, and terms of the option
  - Particularly high relative to option value for short-term options

Definitions of delta and gamma

Delta δ_t: rate at which option value changes with underlying asset price

$$\delta_t \equiv \frac{\partial f(S_t)}{\partial S_t}$$

- $0 < \delta_t < 1$ for plain-vanilla call option
- $-1 < \delta_t < 0$, for plain-vanilla put

Gamma γ_t: rate at which *delta* changes with underlying asset price

$$\gamma_t \equiv \frac{\partial}{\partial S_t}\delta_t = \frac{\partial^2 f(S_t)}{\partial S_t^2}$$

- $\gamma_t \geq 0$ for a vanilla put or call option

The delta-gamma approximation

- Approximate change in value $f(S_t)$ of option on 1 unit of underlying asset—or any security—if S_t changes by ΔS:

$$f(S_t + \Delta S) - f(S_t) \approx \delta_t \Delta S + \frac{1}{2}\gamma_t \Delta S^2$$

  - With other market variables—volatility, risk-free rate, cash flow rate—held constant
- For vanilla option, $f(S_t)$ can represent **Black-Scholes formula**
  - S_t the underlying price
  - With implied volatility, risk-free rate and cash flow rate (dividends, foreign interest, etc.) held constant
- Many other securities have nonlinear responses to changes in a risk factor that can be described similarly
  - For example, bond value can be represented by first- and second-order sensitivities to interest rates

Nonlinearity in market risk

Market risk measurement for options
Applying delta-gamma to the value of an option
Nonlinearity and option risk

Portfolio VaR

Market risk in insurance

Approximating the option return distribution

- Extension of parametric normal VaR approach
  - ΔS—change in underlying price—same as in parametric normal VaR
  - But $f(S_t + \Delta S) - f(S_t)$ the change in option value
- Assume arithmetic returns normally distributed:

$$\frac{\Delta S}{S_t} \sim \mathcal{N}(0, \sigma^2 \tau)$$

  - Estimate return volatility σ of underlying price
- Quantile $z_p \sigma \sqrt{\tau}$ represents scenario for future underlying price return with probability p
- $(1 - p)$-quantile of change in option value approximated by

$$\delta_t z_{1-p} \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_{1-p} \sigma \sqrt{\tau} S_t)^2 \qquad \text{for long call option}$$

$$\delta_t z_p \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_p \sigma \sqrt{\tau} S_t)^2 \qquad \text{for long put option}$$

Delta-gamma VaR for option positions

- Value of an option position: $xf(S_t)$, with x the number of options
  - Apply $x > 0$ for a long option position, $x < 0$ for a short position
  - Use appropriate signs for δ_t and γ_t for P&L of put and call
- τ-period VaR at confidence level α for a *long* option position:
  - Unhedged long call or short put suffers losses when S_t *falls*$\rightarrow$use $z_{1-\alpha}$

$$\mathrm{VaR}_t(\alpha,\tau) = -x\left[\delta_t z_{1-\alpha}\sigma\sqrt{\tau}S_t + \frac{1}{2}\gamma_t\left(z_{1-\alpha}\sigma\sqrt{\tau}S_t\right)^2\right]$$

$$0 < \delta_t < 1, \gamma_t \geq 0, x > 0 \qquad \text{for a long call position}$$

  - Unhedged long put or short call suffers losses from *higher* $S_t$$\rightarrow$use z_α

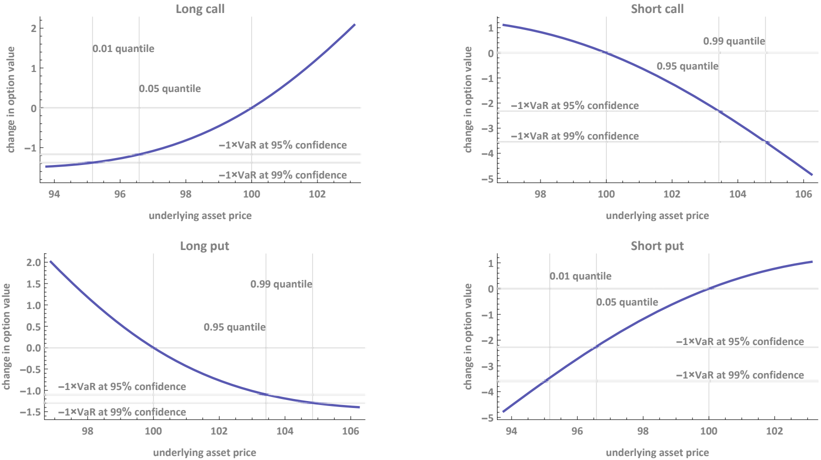
$$\mathrm{VaR}_t(\alpha,\tau) = -x\left[\delta_t z_\alpha\sigma\sqrt{\tau}S_t + \frac{1}{2}\gamma_t(z_\alpha\sigma\sqrt{\tau}S_t)^2\right]$$

$$-1 < \delta_t < 0, \gamma_t \geq 0, x > 0 \qquad \text{for a long put position}$$

Nonlinearity and option risk

- Underlying price moves amplifies loss for long call or short put
- $\delta_t \geq 0$ for a long call, $\delta_t \leq 0$ for a long put, so
  - Unhedged long call and short put positions behave like long positions in underlying
  - Unhedged short call and long put positions behave like short positions in underlying
  - Large-magnitude δ_t increases VaR for a long option position
- $\gamma_t \geq 0$ for a long call or put, so
  - Gamma dampens P&L for long option positions and amplifies P&L for short option positions
  - High γ_t reduces VaR for a long option position and increases VaR for a short option position
  - Difference between P&L results of large and very large underlying price changes is also greater for short positions

Nonlinearity and option risk



Each panel plots the P&L in currency units of an unhedged option position, using the Black-Scholes valuation formula.

Example of delta and gamma calculations

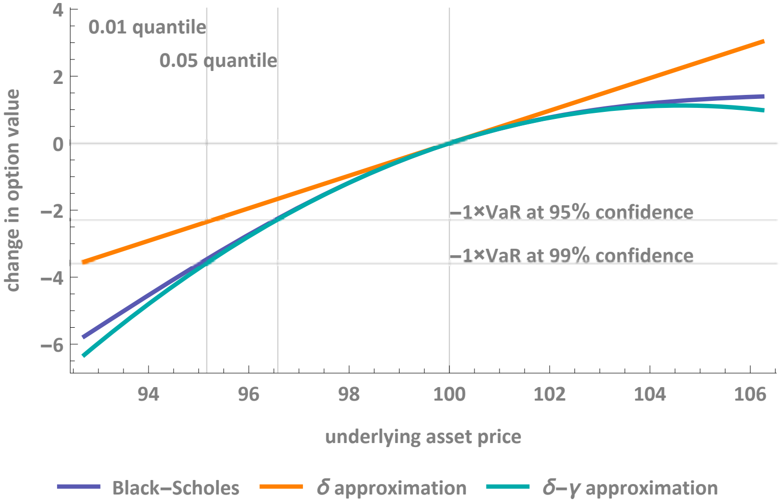
- Short position in at-the-money (ATM) put on one share of non-dividend paying stock with one month to expiry
  - Initial stock price $S_t = 100$, money market rate 1 percent, implied volatility 15 percent
  - Short position, so reverse signs of δ_t and γ_t
- Model of the underlying price: assume zero drift, lognormal returns
- Assume volatility estimate/forecast 15 percent per annum, equal to implied vol
  - But note historical volatility estimate generally somewhat lower than implied vol (**volatility premium**)
- To compute one-week VaR ($\tau = \frac{1}{52}$), compare option value at initial underlying price to value in VaR scenario
  - P&L: value of 3-week options with shock to underlying price minus value without shock
  - Excludes time decay—which is non-random—from revaluation
  - But retain zero-drift assumption on underlying price

Delta-gamma VaR results

- Black-Scholes delta of 3-week ATM put is -0.4857; gamma is 0.1050
  - Short put has delta equivalent of 48.50 worth of stock
  - And high gamma: e.g. delta declines to -0.3829 if price rises to 101
- Compute VaR scenarios—quantiles of $S_{t+\tau}$ for $\alpha = 0.95, 0.99$
  - With $\sigma = 0.15$ annually, $\sigma\sqrt{\tau} = 0.0208$
- Delta-gamma results in good approximation for non-extreme changes in S_t
- Compare VaR computed using Black-Scholes formula, changing only underlying price

		VaR estimates		
	VaR scenario	delta-only	delta-gamma	Black-Scholes
$\alpha = 0.95$	-3.422	1.662	2.276	2.250
$\alpha = 0.99$	-4.839	2.350	3.579	3.465

Delta and delta-gamma approximations



Short put option struck at 100, initial underlying asset price 100, money market rate 1 percent, valued using Black-Scholes formula.

Nonlinearity in market risk

Market risk measurement for options

Portfolio VaR

- Algebra of portfolio VaR
- Example of portfolio VaR
- Delta-normal approach to VaR computation

Market risk in insurance

Most VaR applications involve portfolios

- Multiple risk factors and/or multiple positions, e.g.
  - Hedged positions
  - Relative value trades such as spread trades
  - More general portfolios of long and short positions
  - Portfolio products such as structured credit
- Introduces additional complications to convexity:
  - Need to take account of *correlations* of risk factor returns
- May have P&L that is **nonmonotone** with respect to a risk factor's returns
  - *Sign* of $\frac{\partial f(S_t)}{\partial S_t}$ may change with S_t
- Example of nonmonotonicity: delta-hedged options, exposed to gamma
  - Long gamma: largest losses for smallest underlying returns
- **Delta-normal**: simple approach to computing portfolio VaR for market risk
  - But may be drastically inaccurate for some portfolios, e.g. delta-hedged options

Parametric computation of portfolio VaR

- Apply algebra of portfolio returns to sequence of computations of parametric single-position VaR
- Assume logarithmic risk factor returns jointly normal

$$\mathbf{r}_t = (r_{1,t}, r_{2,t}, \ldots, r_{n,t})'$$

- Risk factor returns have time-varying variance-covariance matrix $\boldsymbol{\Sigma}_t$
- Portfolio volatility with portfolio weights on risk factors an n-dimensional vector $\mathbf{w}$:

$$\sigma_t = \mathbf{w}'\boldsymbol{\Sigma}_t\mathbf{w}$$

- VaR in return terms at confidence level α equal to $z_\alpha \sigma_t \sqrt{\tau}$

Estimating the covariance matrix

- Compute volatilities and correlations of the n risk factors from the variances and covariances constituting $\mathbf{\Sigma}_t$
- Can be estimated via EWMA, with a decay factor λ, via

$$\begin{aligned}\mathbf{\Sigma}_t &= \frac{1-\lambda}{1-\lambda^m}\sum_{\tau=1}^{m}\lambda^{m-\tau}\mathbf{r}_t'\mathbf{r}_t \\ &\approx \lambda\mathbf{\Sigma}_{t-1} + (1-\lambda)\mathbf{r}_t'\mathbf{r}_t\end{aligned}$$

  - $\mathbf{r}_t'\mathbf{r}_t$ an outer product of return vector on date t
  - Square matrix with same dimension as $\mathbf{\Sigma}_t$
- VaR in return terms at confidence level α equal to $z_\alpha \sigma_t \sqrt{\tau}$

Two-position portfolio

- Two positions or risk factors: 3 parameters to estimate

$$\boldsymbol{\Sigma}_t = \begin{pmatrix} \sigma_{1,t}^2 & \sigma_{1,t}\sigma_{2,t}\rho_{12,t} \\ \sigma_{1,t}\sigma_{2,t}\rho_{12,t} & \sigma_{2,t}^2 \end{pmatrix}$$

- Return volatility of a two-position portfolio

$$\sigma_t^2 = w_1^2\sigma_{1,t}^2 + w_2^2\sigma_{2,t}^2 + 2w_1w_2\sigma_{1,t}\sigma_{2,t}\rho_{12,t}$$

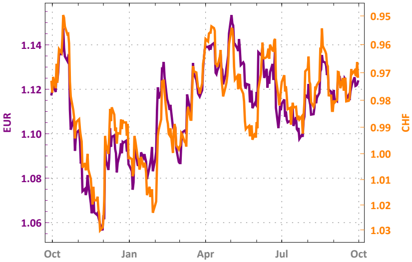
Long-short currency trade

- Long EUR and short CHF against USD, potentially motivated by view that
    - Extremely sharp safe-haven appreciation of CHF relative to EUR since beginning of global financial crisis economically unsustainable
    - "Risk-on" strategy: global recovery, decrease in uncertainty and risk aversion will reverse CHF appreciation
- Weights are 1 and -1
- Measure of risk at time t is

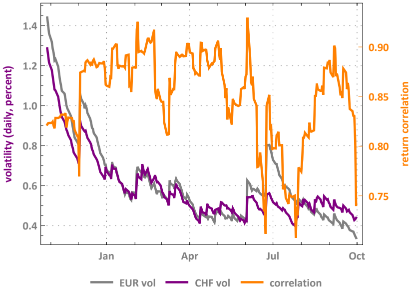
$$(1, -1)\,\boldsymbol{\Sigma}_t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sigma_{1,t}^2 + \sigma_{2,t}^2 - 2\sigma_{1,t}\sigma_{2,t}\rho_{12,t}$$

- VaR expressed as quantile of USD portfolio loss relative to market value of one side of the trade

EUR-USD and USD-CHF risk parameters 2015-2016

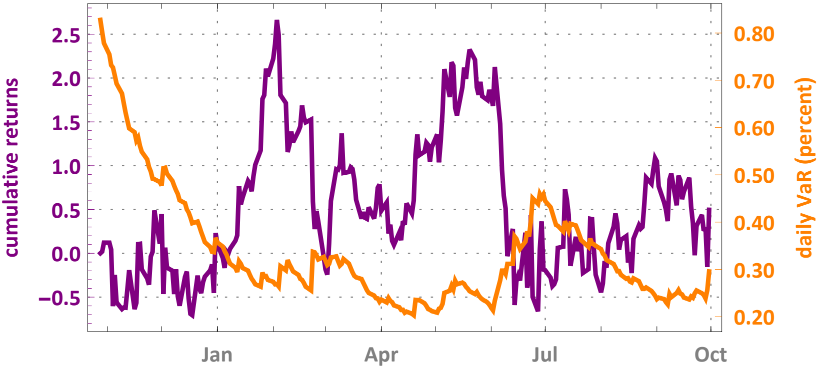


EUR-USD and USD-CHF exchange rates, daily, 30Sep2015 to 30Sep2016. USD-CHF rates on an inverted scale.



Return volatilities and correlation of EUR-USD and USD-CHF exchange rates, daily, 28Oct2015 to 30Sep2016. Estimated via EWMA with decay factor $\lambda = 0.94$.

Long EUR-USD versus short USD-CHF risk and returns 2015-2016



Cumulative returns on a portfolio consisting of a long position in EUR and position in CHF against USD, daily, 30Sep2015 to 30Sep2016.

Delta-normal VaR

- **Delta-normal VaR:** form of parametric VaR
- Simplification of VaR by means of two approximations:
  - Linearize exposures to risk factors
  - Treat arithmetic, not log returns, as normally distributed
- Letting $f(S_t)$ now represent the value of a security not necessarily an option, delta δ_t defined as the derivative or value w.r.t. risk factor:

$$\delta_t \equiv \frac{\partial f(S_t)}{\partial S_t}$$

  - δ_t may be positive or negative, > 1 in magnitude
  - How many deltas and how they are measured depend on modeling choices: S_t may be a vector
- Limitations: doesn't capture convexity, other non-linearities

Delta equivalents

- **Delta equivalent** $x\delta_t S_t$ of a position
  - Or $\delta_t S_t$ per unit
- Measure of exposure, states how position affected by unit underlying risk factor return
  - Delta equivalent plays crucial role in hedging option risk
  - At underlying price S_t, position of x options with δ_t has same response to small price change as underlying position $x\delta_t S_t$

Delta-normal VaR for a single position

- In many cases $\delta_t = \pm 1$
    - If risk factor identical to the security
        - Often the case for major foreign currencies, equity indexes
        - $\delta_t = -1$ for short position
    - Value of a security varies one-for-one with risk factor
        - E.g. local currency value of foreign stock as function of exchange rate
- Delta-normal VaR for a single position exposed to single risk factor at confidence level α:

$$\mathrm{VaR}_t(\alpha, \tau)(x) = -z_{1-\alpha}\sigma\sqrt{\tau}x\delta_t S_t$$

    - Identical to approximation for single long position parametric VaR
    - For short position, uses $z_{1-\alpha}$ rather than z_α, offset by $\delta_t = -1$
    - Normality rather than lognormality of returns⇒long and short positions have identical VaR
- Single position exposed to several risk factors (→portfolio VaR)

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Annuities and market risk

Inflation risk

Types of annuities

- **Annuities** are contracts for exchange of a specified sequence of payments between an **annuitant** and intermediary, generally an insurance company
- Very wide variety of types
- Payments by annuitant may be a lump sum or periodic over a future time interval
  - Annuities with periodic future payments may **lapse** or include **early surrender penalties**
- Payments by insurance company may be fixed or vary:

  Fixed annuity: payments or interest rate fixed over time

  Variable annuity: payments vary with return on a specified portfolio, generally equity-focused
- Annuities may include guarantees by insurance company, such as **guaranteed minimum benefits**

Risks of annuity issuance

- Market risks interact with risks arising from guarantees and policyholder behavior
- Variable annuities generally provide guaranteed minimum return
  - Economically equivalent to sale of put option on equity market by insurer to policyholder
  - Annuity is underpriced if value of put not fully incorporated
- Large losses to U.S. insurers in 2008
  - Hartford Life became a Troubled Asset Relief Program (TARP) recipient
- Fixed annuity issuance exposed to **convexity risk**
  - Assets generally duration-matched to liabilities
  - But liabilities exhibit greater convexity due to guarantees and policyholder behavior
  - Economically equivalent to sale of put option on bond market by insurer to policyholder
- Rising interest rates: early surrender optimal $\rightarrow$ duration falls rapidly
- Falling interest rates: minimum guaranteed rate in effect $\rightarrow$ duration rises rapidly

Inflation risk

- **Inflation rate risk** is the risk of loss from a rise in the general price level
    - Directly affects securities with payoffs defined in nominal terms
    - Indirectly affects real assets by affecting macroeconomic conditions
- Inflation difficult to hedge
    - **Inflation-indexed bonds** have yields defined in real terms
    - **Inflation swaps** and other derivatives

Insurance company exposure to inflation

- Insurers may benefit from inflation
- Long-term liabilities generally defined in nominal terms
    - Generally not fully hedged against changes in interest rates
    - And substantial allocation to real assets: real estate, equities
- Permanent risk in inflation rate reduces real value of liabilities