

Lecture notes on money, finance, risk management, and public policy

# Market risk measurement in practice

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Nonlinearity in market risk

Market risk measurement for options

Portfolio VaR

Market risk in insurance

## **Nonlinearity in market risk**

Nonlinearity and convexity

Delta-gamma and option risk

Measuring Value-at-Risk for a bond position

## Market risk measurement for options

## Portfolio VaR

## Market risk in insurance

# Nonlinearity in market risk

- **Nonlinearity:** P&L or payoff of a security doesn't respond proportionally to risk factor returns
- **Examples** of securities with nonlinear payoffs:
  - Options: **gamma risk**
  - Bonds: **convexity risk** in non-callable as well as callable coupon and zero-coupon, mortgage-backed securities
- Security value  $f(S_t)$  a function of risk factor  $S_t$ ,
- $f(S_t)$  has second and nonzero higher derivatives→
  - Large-magnitude returns have a proportionally larger or smaller P&L impact than small returns
  - →Price changes in one direction may have a larger P&L impact than changes in the opposite direction

# VaR techniques for nonlinear positions

**Simulation with full repricing** using asset valuation model, e.g. Black-Scholes formula

- Can use Monte Carlo or historical simulation of **underlying price** or risk factor returns
- But revaluation of position at each simulated return may itself require “expensive” simulations

**Delta-gamma** using linear-quadratic approximation of P&L responses to risk factor returns

- Trades accuracy for speed
- Tractable and quite accurate in many cases
- But may be inaccurate for some portfolios
- Can be combined with Monte Carlo or historical simulation of risk factor returns

## Option risk and the “greeks”

- Option risk stemming from underlying asset price risk is nonlinear
  - Price risk of the underlying asset (→ **delta**, **gamma**)
  - Sensitivity to underlying price greatest near strike, may fall off rapidly in- or out-of-the-money
  - → Apply delta-gamma, with  $f(S_t)$  representing option price or value
- Options are exposed to other risk factors, including
  - Interest-rate risk or **rho**, since an option matures at a future date
  - Implied volatility or **vega** risk
- Options have **time value** that decays over time at a rate **theta**
  - Theta is not a risk, but a deterministic quantity
  - Depends on interest rates, implied volatility, and terms of the option
  - Particularly high relative to option value for short-term options

## Definitions of delta and gamma

**Delta**  $\delta_t$ : rate at which option value changes with underlying asset price

$$\delta_t \equiv \frac{\partial f(S_t)}{\partial S_t}$$

- $0 < \delta_t < 1$  for plain-vanilla call option
- $-1 < \delta_t < 0$ , for plain-vanilla put

**Gamma**  $\gamma_t$ : rate at which *delta* changes with underlying asset price

$$\gamma_t \equiv \frac{\partial}{\partial S_t} \delta_t = \frac{\partial^2 f(S_t)}{\partial S_t^2}$$

- $\gamma_t \geq 0$  for a vanilla put or call option

## The delta-gamma approximation

- Approximate change in value  $f(S_t)$  of option on 1 unit of underlying asset—or any security—if  $S_t$  changes by  $\Delta S$ :

$$f(S_t + \Delta S) - f(S_t) \approx \delta_t \Delta S + \frac{1}{2} \gamma_t \Delta S^2$$

- With other market variables—volatility, risk-free rate, cash flow rate—held constant
- For vanilla option,  $f(S_t)$  can represent **Black-Scholes formula**
  - $S_t$  the underlying price
  - With implied volatility, risk-free rate and cash flow rate (dividends, foreign interest, etc.) held constant
- Many other securities have nonlinear responses to changes in a risk factor that can be described similarly
  - For example, bond value can be represented by first- and second-order sensitivities to interest rates

## Quantiles of the distribution of yield

- Value-at-Risk (VaR) measure requires interest rate volatility estimate
- Assume yield lognormally distributed
  - $\Rightarrow$  Proportional yield changes normally distributed
- Compute return quantiles using yield volatility estimate, appropriate normal quantile, and time horizon  $\tau$ :
  - $z_\alpha$  :  $\alpha$ -quantile of standard normal distribution
  - $\sigma_{y,t}$ : statistical or risk-neutral estimate of itemize volatility
- Estimated  $\alpha$ -quantile of proportional change in yield from  $t$  to  $t + \tau$  is then  $e^{z_\alpha \sigma_t \sqrt{\tau}} - 1$ 
  - Arithmetic approximation:  $z_\alpha \sigma_{y,t}$
- Multiply VaR quantile by yield to obtain yield changes in basis points  $y_t \left( e^{z_\alpha \sigma_t \sqrt{\tau}} - 1 \right)$ 
  - Equivalent to using normal vol, hence the name

$$z_\alpha \sigma_{y,t} \sqrt{\tau} y_t = z_\alpha \sigma_{n,t} \sqrt{\tau}$$

## Computing VaR for a bond

- If the model is accurate, the probability is  $\alpha$  that yield will rise (fall) no more than  $100z_\alpha\sigma_{y,t}$  ( $100z_{1-\alpha}\sigma_{y,t}$ ) percent
- Use upper (lower) tail of normal distribution for long (short) bond position
  - Since bond value declines as rates rise
- Apply ( $\rightarrow$ )delta-gamma approach, with duration and convexity in roles of delta and gamma
  - Set  $\Delta y$  to the VaR quantile of the change in yield in bps
- VaR—in percent of value—of a long position is then (using normal vol)

$$mdur_t z_\alpha \sigma_{n,t} \sqrt{\mathcal{T}} - \frac{1}{2} conv_t (z_\alpha \sigma_{n,t} \sqrt{\mathcal{T}})^2$$

- The signs are reversed to convert the negative impact of a rise in yield to a positive number

## Example: VaR of long U.S. 10-year note position

- Assume yield (0.023381 in example) lognormally distributed, proportional yield changes then normal
- Use ballpark estimate of normal vol drawn from option markets—90 bps per annum—to compute yield volatility and VaR shock
  - $\Rightarrow$ Yield volatility  $\frac{0.0090}{0.023381} = 0.384928$  or 38.5 percent at annual rate
- One-day return quantile of order 0.01 (99 percent confidence level):
  - Divide annual vol by  $\sqrt{256} = 16$ , multiply by  $z_{\alpha} = 2.32635$
  - $\Rightarrow$ Yield volatility  $2.32635 \frac{0.384928}{16} = 0.0559673$  or a 5.6 percent change in yield
  - VaR quantile in basis points is  $2.32635 \frac{0.384928}{16} \cdot 0.023381 = 2.32635 \frac{0.0090}{16} = 0.00130857$
- VaR in dollars: initial bond value times proportional change in value using pertinent VaR yield change quantile

$$\left( 8.816 \cdot 0.00131 - \frac{1}{2} 87.13 \cdot 0.00131^2 \right) \times 970\,468.75 = 11\,123.30$$

## Nonlinearity in market risk

### **Market risk measurement for options**

Applying delta-gamma to the value of an option

Nonlinearity and option risk

## Portfolio VaR

## Market risk in insurance

## Approximating the option return distribution

- Extension of parametric normal VaR approach
  - $\Delta S$ —change in underlying price—same as in parametric normal VaR
  - But  $f(S_t + \Delta S) - f(S_t)$  the change in option value
- Assume arithmetic returns normally distributed:

$$\frac{\Delta S}{S_t} \sim \mathcal{N}(0, \sigma^2 \tau)$$

- Estimate return volatility  $\sigma$  of underlying price
- Quantile  $z_p \sigma \sqrt{\tau}$  represents scenario for future underlying price return with probability  $p$
- $(1 - p)$ -quantile of change in option value approximated by

$$\delta_t z_{1-p} \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_{1-p} \sigma \sqrt{\tau} S_t)^2 \quad \text{for long call option}$$

$$\delta_t z_p \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_p \sigma \sqrt{\tau} S_t)^2 \quad \text{for long put option}$$

## Delta-gamma VaR for option positions

- Value of an option position:  $x f(S_t)$ , with  $x$  the number of options
  - Apply  $x > 0$  for a long option position,  $x < 0$  for a short position
  - Use appropriate signs for  $\delta_t$  and  $\gamma_t$  for P&L of put and call
- $\tau$ -period VaR at confidence level  $\alpha$  for a *long* option position:
  - Unhedged long call or short put suffers losses when  $S_t$  falls  $\rightarrow$  use  $z_{1-\alpha}$

$$\text{VaR}_t(\alpha, \tau) = -x \left[ \delta_t z_{1-\alpha} \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_{1-\alpha} \sigma \sqrt{\tau} S_t)^2 \right]$$

$$0 < \delta_t < 1, \gamma_t \geq 0, x > 0 \quad \text{for a long call position}$$

- Unhedged long put or short call suffers losses from *higher*  $S_t \rightarrow$  use  $z_\alpha$

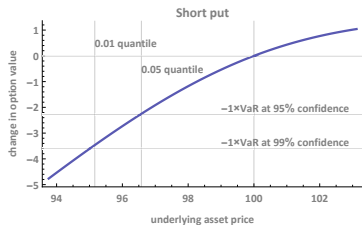
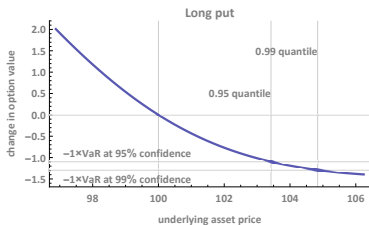
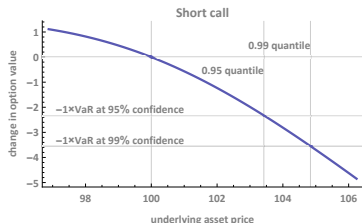
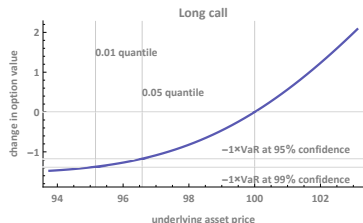
$$\text{VaR}_t(\alpha, \tau) = -x \left[ \delta_t z_\alpha \sigma \sqrt{\tau} S_t + \frac{1}{2} \gamma_t (z_\alpha \sigma \sqrt{\tau} S_t)^2 \right]$$

$$-1 < \delta_t < 0, \gamma_t \geq 0, x > 0 \quad \text{for a long put position}$$

# Nonlinearity and option risk

- Underlying price moves amplifies loss for long call or short put
- $\delta_t \geq 0$  for a long call,  $\delta_t \leq 0$  for a long put, so
  - Unhedged long call and short put positions behave like long positions in underlying
  - Unhedged short call and long put positions behave like short positions in underlying
  - Large-magnitude  $\delta_t$  increases VaR for a long option position
- $\gamma_t \geq 0$  for a long call or put, so
  - Gamma dampens P&L for long option positions and amplifies P&L for short option positions
  - High  $\gamma_t$  reduces VaR for a long option position and increases VaR for a short option position
  - Difference between P&L results of large and very large underlying price changes is also greater for short positions

# Nonlinearity and option risk



Each panel plots the P&L in currency units of an unhedged option position, using the Black-Scholes valuation formula.

## Example of delta and gamma calculations

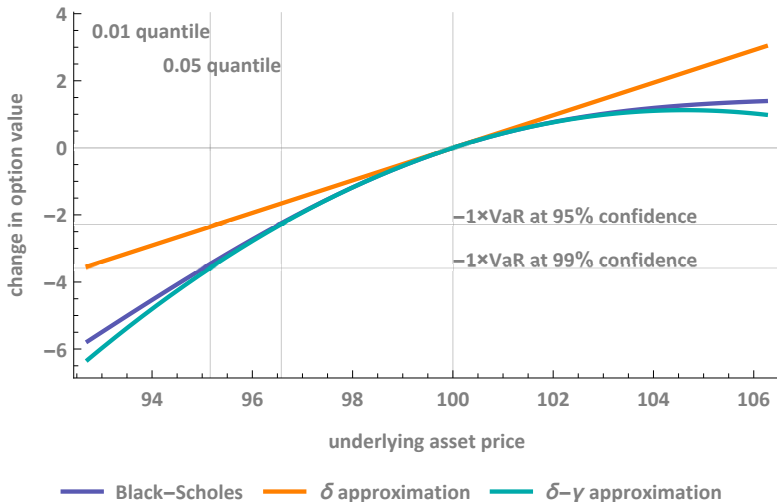
- Short position in at-the-money (ATM) put on one share of non-dividend paying stock with one month to expiry
  - Initial stock price  $S_t = 100$ , money market rate 1 percent, implied volatility 15 percent
  - Short position, so reverse signs of  $\delta_t$  and  $\gamma_t$
- Model of the underlying price: assume zero drift, lognormal returns
- Assume volatility estimate/forecast 15 percent per annum, equal to implied vol
  - But note historical volatility estimate generally somewhat lower than implied vol ( **volatility premium** )
- To compute one-week VaR ( $\tau = \frac{1}{52}$ ), compare option value at initial underlying price to value in VaR scenario
  - P&L: value of 3-week options with shock to underlying price minus value without shock
  - Excludes time decay—which is non-random—from revaluation
  - But retain zero-drift assumption on underlying price

## Delta-gamma VaR results

- Black-Scholes delta of 3-week ATM put is -0.4857; gamma is 0.1050
  - Short put has delta equivalent of 48.50 worth of stock
  - And high gamma: e.g. delta declines to -0.3829 if price rises to 101
- Compute VaR scenarios—quantiles of  $S_{t+\tau}$  for  $\alpha = 0.95, 0.99$ 
  - With  $\sigma = 0.15$  annually,  $\sigma\sqrt{\tau} = 0.0208$
- Delta-gamma results in good approximation for non-extreme changes in  $S_t$
- Compare VaR computed using Black-Scholes formula, changing only underlying price

	VaR scenario	VaR estimates		
		delta-only	delta-gamma	Black-Scholes
$\alpha = 0.95$	-3.422	1.662	2.276	2.250
$\alpha = 0.99$	-4.839	2.350	3.579	3.465

## Delta and delta-gamma approximations



Short put option struck at 100, initial underlying asset price 100, money market rate 1 percent, valued using Black-Scholes formula.

Nonlinearity in market risk

Market risk measurement for options

## **Portfolio VaR**

- Algebra of portfolio VaR

- Example of portfolio VaR

- Delta-normal approach to VaR computation

Market risk in insurance

## Most VaR applications involve portfolios

- Multiple risk factors and/or multiple positions, e.g.
  - Hedged positions
  - Relative value trades such as spread trades
  - More general portfolios of long and short positions
  - Portfolio products such as structured credit
- Introduces additional complications to convexity:
  - Need to take account of *correlations* of risk factor returns
- May have P&L that is **nonmonotone** with respect to a risk factor's returns
  - *Sign* of  $\frac{\partial f(S_t)}{\partial S_t}$  may change with  $S_t$
- Example of nonmonotonicity: delta-hedged options, exposed to gamma
  - Long gamma: largest losses for smallest underlying returns
- **Delta-normal**: simple approach to computing portfolio VaR for market risk
  - But may be drastically inaccurate for some portfolios, e.g. delta-hedged options

# Parametric computation of portfolio VaR

- Apply algebra of portfolio returns to sequence of computations of parametric single-position VaR
- Assume logarithmic risk factor returns jointly normal

$$\mathbf{r}_t = (r_{1,t}, r_{2,t}, \dots, r_{n,t})'$$

- Risk factor returns have time-varying variance-covariance matrix  $\boldsymbol{\Sigma}_t$
- Portfolio volatility with portfolio weights on risk factors an  $n$ -dimensional vector  $\mathbf{w}$ :

$$\sigma_t = \mathbf{w}' \boldsymbol{\Sigma}_t \mathbf{w}$$

- VaR in return terms at confidence level  $\alpha$  equal to  $z_\alpha \sigma_t \sqrt{T}$

## Estimating the covariance matrix

- Compute volatilities and correlations of the  $n$  risk factors from the variances and covariances constituting  $\Sigma_t$
- Can be estimated via EWMA, with a decay factor  $\lambda$ , via

$$\begin{aligned}\Sigma_t &= \frac{1 - \lambda}{1 - \lambda^m} \sum_{\tau=1}^m \lambda^{m-\tau} \mathbf{r}'_t \mathbf{r}_t \\ &\approx \lambda \Sigma_{t-1} + (1 - \lambda) \mathbf{r}'_t \mathbf{r}_t\end{aligned}$$

- $\mathbf{r}'_t \mathbf{r}_t$  an outer product of return vector on date  $t$
- Square matrix with same dimension as  $\Sigma_t$
- VaR in return terms at confidence level  $\alpha$  equal to  $z_\alpha \sigma_t \sqrt{T}$

## Two-position portfolio

- Two positions or risk factors: 3 parameters to estimate

$$\Sigma_t = \begin{pmatrix} \sigma_{1,t}^2 & \sigma_{1,t}\sigma_{2,t}\rho_{12,t} \\ \sigma_{1,t}\sigma_{2,t}\rho_{12,t} & \sigma_{2,t}^2 \end{pmatrix}$$

- Return volatility of a two-position portfolio

$$\sigma_t^2 = w_1^2\sigma_{1,t}^2 + w_2^2\sigma_{2,t}^2 + 2w_1w_2\sigma_{1,t}\sigma_{2,t}\rho_{12,t}$$

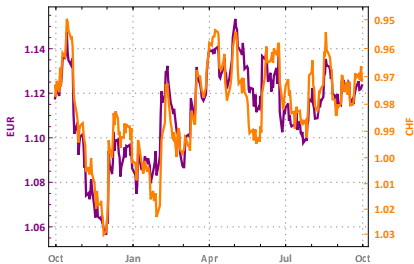
## Long-short currency trade

- Long EUR and short CHF against USD, potentially motivated by view that
  - Extremely sharp safe-haven appreciation of CHF relative to EUR since beginning of global financial crisis economically unsustainable
  - “Risk-on” strategy: global recovery, decrease in uncertainty and risk aversion will reverse CHF appreciation
- Weights are 1 and -1
- Measure of risk at time  $t$  is

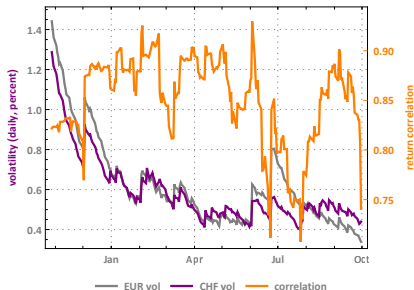
$$(1, -1) \boldsymbol{\Sigma}_t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sigma_{1,t}^2 + \sigma_{2,t}^2 - 2\sigma_{1,t}\sigma_{2,t}\rho_{12,t}$$

- VaR expressed as quantile of USD portfolio loss relative to market value of one side of the trade

# EUR-USD and USD-CHF risk parameters 2015-2016

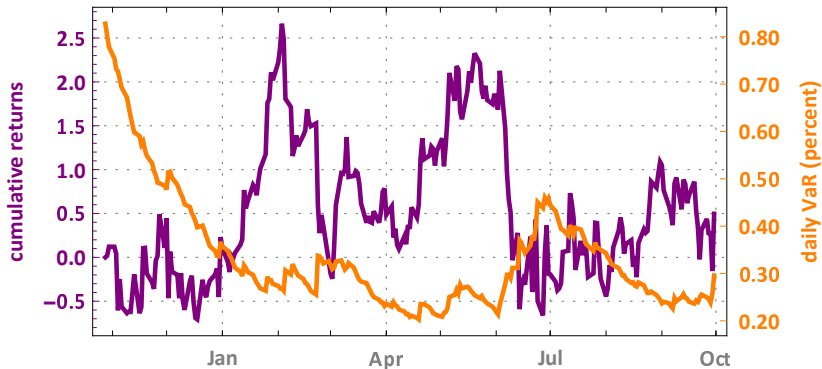


EUR-USD and USD-CHF exchange rates, daily, 30Sep2015 to 30Sep2016. USD-CHF rates on an inverted scale.



Return volatilities and correlation of EUR-USD and USD-CHF exchange rates, daily, 28Oct2015 to 30Sep2016. Estimated via EWMA with decay factor  $\lambda = 0.94$ .

# Long EUR-USD versus short USD-CHF risk and returns 2015-2016



Cumulative returns on a portfolio consisting of a long position in EUR and position in CHF against USD, daily, 30Sep2015 to 30Sep2016.

# Delta-normal VaR

- **Delta-normal VaR:** form of parametric VaR
- Simplification of VaR by means of two approximations:
  - Linearize exposures to risk factors
  - Treat arithmetic, not log returns, as normally distributed
- Letting  $f(S_t)$  now represent the value of a security not necessarily an option, delta  $\delta_t$  defined as the derivative or value w.r.t. risk factor:

$$\delta_t \equiv \frac{\partial f(S_t)}{\partial S_t}$$

- $\delta_t$  may be positive or negative,  $> 1$  in magnitude
  - How many deltas and how they are measured depend on modeling choices:  $S_t$  may be a vector
- Limitations: doesn't capture convexity, other non-linearities

# Delta equivalents

- **Delta equivalent**  $x\delta_t S_t$  of a position
  - Or  $\delta_t S_t$  per unit
- Measure of exposure, states how position affected by unit underlying risk factor return
  - Delta equivalent plays crucial role in hedging option risk
  - At underlying price  $S_t$ , position of  $x$  options with  $\delta_t$  has same response to small price change as underlying position  $x\delta_t S_t$

## Delta-normal VaR for a single position

- In many cases  $\delta_t = \pm 1$ 
  - If risk factor identical to the security
    - Often the case for major foreign currencies, equity indexes
    - $\delta_t = -1$  for short position
  - Value of a security varies one-for-one with risk factor
    - E.g. local currency value of foreign stock as function of exchange rate
- Delta-normal VaR for a single position exposed to single risk factor at confidence level  $\alpha$ :

$$\text{VaR}_t(\alpha, \tau)(x) = -z_{1-\alpha} \sigma \sqrt{\tau} x \delta_t S_t$$

- Identical to approximation for single long position parametric VaR
- For short position, uses  $z_{1-\alpha}$  rather than  $z_\alpha$ , offset by  $\delta_t = -1$
- Normality rather than lognormality of returns  $\Rightarrow$  long and short positions have identical VaR
- Single position exposed to several risk factors ( $\rightarrow$  portfolio VaR)

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**Market risk in insurance**

Annuities and market risk

Inflation risk

# Types of annuities

- **Annuities** are contracts for exchange of a specified sequence of payments between an **annuitant** and intermediary, generally an insurance company
- Very wide variety of types
- Payments by annuitant may be a lump sum or periodic over a future time interval
  - Annuities with periodic future payments may **lapse** or include **early surrender penalties**
- Payments by insurance company may be fixed or vary:
  - Fixed annuity:** payments or interest rate fixed over time
  - Variable annuity:** payments vary with return on a specified portfolio, generally equity-focused
- Annuities may include guarantees by insurance company, such as **guaranteed minimum benefits**

## Risks of annuity issuance

- Market risks interact with risks arising from guarantees and policyholder behavior
- Variable annuities generally provide guaranteed minimum return
  - Economically equivalent to sale of put option on equity market by insurer to policyholder
  - Annuity is underpriced if value of put not fully incorporated
- Large losses to U.S. insurers in 2008
  - Hartford Life became a Troubled Asset Relief Program (TARP) recipient
- Fixed annuity issuance exposed to **convexity risk**
  - Assets generally duration-matched to liabilities
  - But liabilities exhibit greater convexity due to guarantees and policyholder behavior
  - Economically equivalent to sale of put option on bond market by insurer to policyholder
- Rising interest rates: early surrender optimal → duration falls rapidly
- Falling interest rates: minimum guaranteed rate in effect → duration rises rapidly

# Inflation risk

- **Inflation rate risk** is the risk of loss from a rise in the general price level
  - Directly affects securities with payoffs defined in nominal terms
  - Indirectly affects real assets by affecting macroeconomic conditions
- Inflation difficult to hedge
  - **Inflation-indexed bonds** have yields defined in real terms
  - **Inflation swaps** and other derivatives

# Insurance company exposure to inflation

- Insurers may benefit from inflation
- Long-term liabilities generally defined in nominal terms
  - Generally not fully hedged against changes in interest rates
  - And substantial allocation to real assets: real estate, equities
- Permanent risk in inflation rate reduces real value of liabilities